On monodromy in integrable Hamiltonian systems

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Chapter 3

Fractional monodromy and Seifert manifolds

The notion of fractional monodromy was introduced by Nekhoroshev, Sadovskií and Zhilinskií as a generalization of Hamiltonian ('integer') monodromy in the sense of Duistermaat from torus bundles to singular torus fibrations. In the present chapter we generalize the results obtained in Chapters 1 and 2 to this singular case and make a connection to Fomenko-Zieschang theory. In particular, we show that fractional monodromy can be described in terms of the Euler number of an appropriately chosen Seifert fibration and that this number can be computed in terms of the fixed points of the circle action.

3.1 Parallel transport and fractional monodromy

As we have seen in the previous chapters, Hamiltonian monodromy is intimately related to the singularities of a given integrable system. However, this invariant is defined for the non-singular part $F\colon F^{-1}(R) \to R$ of the possibly singular torus fibration $F\colon M \to \mathbb{R}^n$ that comes with the system. An invariant that generalizes Hamiltonian monodromy to singular torus fibrations was introduced in [79] and it is called *fractional monodromy.* Before defining this invariant, let us make a few preliminary remarks.

In this chapter, we work in the setting of *singular Lagrangian fibrations*, which is slightly more general than the setting of integrable systems.

Definition 3.1.1. Let M be a symplectic manifold and B be a manifold of half the dimension of M. A smooth map $F\colon M \to B$ will be called a (*singular*) *Lagrangian fibration* if for almost all $x \in M$, we have that $\mathrm{Ker}(dF_x)$ is a Lagrangian subspace[1] of T_xM. Note that, in this case, the differential dF_x is surjective.

[1] A subspace W of a symplectic vector space V is called *Lagrangian* if it is *isotropic*, that is, if the symplectic form vanishes on W, and of maximal dimension $\dim W = \dim V/2$.

Proposition 3.1.2. (Weinstein, [102]) *An integral map defines a (possibly singular) Lagrangian fibration. Conversely, for a (singular) Lagrangian fibration $F\colon M \to B$ and a chart (U, χ) of B, the composition map $F \circ \chi$ defines an integrable system on $F^{-1}(U) \subset M$.*

Proof. In the non-singular case, we have that all the fibers $F^{-1}(\xi)$ are Lagrangian, and the result is proven in [102]. The singular case is similar. □

It follows from Proposition 3.1.2 that one can define Hamiltonian monodromy of a Lagrangian torus bundle essentially in the same way as in Chapter 1.

Specifically, consider a Lagrangian torus bundle $F\colon M \to B$. Let T^*B be the cotangent bundle of B. Observe that there is an action of the cotangent spaces of T^*B on the fibers of the bundle F, which, in every chart (V, χ) is given by the $\mathbb{R}^n$ action of $F \circ \chi$. For each $\xi \in B$, the stabilizer of this action on the fiber $F^{-1}(\xi)$ is a lattice $\mathbb{Z}^n_\xi \subset T^*_\xi B$. Since

$$F^{-1}(\xi) \simeq T^*_\xi B / \mathbb{Z}^n_\xi,$$

the lattice $\mathbb{Z}^n_\xi$ can be identified with the first integer homology group $H_1(F^{-1}(\xi))$. The union of these lattices gives rise to the covering

$$\mathrm{Pr}\colon P = \bigcup \mathbb{Z}^n_\xi \to B.$$

The notions of parallel transport of homology cycles and Hamiltonian monodromy can be defined in terms of this covering as follows.

Definition 3.1.3. Let $\gamma\colon [0,1] \to B$ be a continuous path and $c \in H_1(F^{-1}(\gamma(0)))$ be a homology cycle. Let $\tilde{\gamma}$ be the lift of γ to the covering space P that starts at c. The *parallel transport* of c along γ is the homology cycle $\tilde{\gamma}(1) \in H_1(F^{-1}(\gamma(1)))$.

Definition 3.1.4. (Duistermaat [27]) The *Hamiltonian monodromy* of the bundle F is defined as the automorphism of $\mathbb{Z}^n_{\xi_0} \simeq H_1(F^{-1}(\xi_0))$ that is induced by the parallel transport.

In order to define fractional monodromy, we need the following generalized version of parallel transport, which is essentially due to [35, 51].

Definition 3.1.5. Let X be a manifold with the boundary ∂X consisting of two connected components X_0 and X_1. The cycle $\alpha_1 \in H_1(X_1)$ is a *parallel transport* of the cycle $\alpha_0 \in H_1(X_0)$ *along* X if

$$(\alpha_0, -\alpha_1) \in \partial_*(H_2(X, \partial X)),$$

where ∂_* is the connecting homomorphism of the exact sequence

$$\cdots \to H_2(X) \to H_2(X, \partial X) \xrightarrow{\partial_*} H_1(\partial X) \to H_1(X) \to \cdots$$

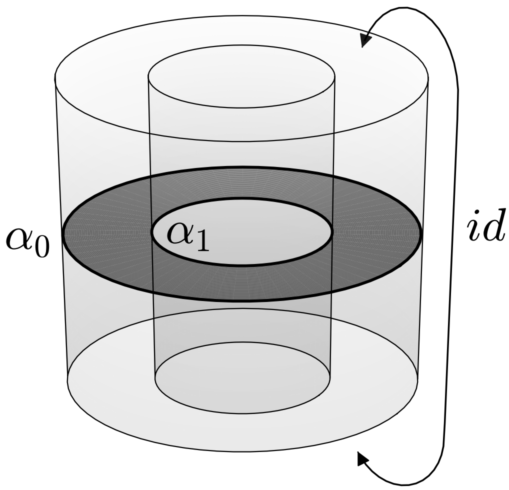

Figure 3.1: Parallel transport along X.

Remark 3.1.6. For compact 3 manifolds Definition 3.1.5 can be reformulated as follows (see [52]): α_1 is a parallel transport of α_0 along X if there exists an oriented 2-dimensional submanifold $S \subset X$ that 'connects' α_0 and α_1:

$$\partial S = S_0 \sqcup S_1 \text{ and } [S_i] = (-1)^i \alpha_i \in H_1(X_i);$$

see Fig. 3.1. We note, however, that even for compact 3-manifolds it might happen that, for a given homology cycle, the parallel transport is not defined or is not unique. As we shall show later, for Seifert manifolds (with an orientable base) the parallel transport is unique; see Theorem 3.3.5.

The following lemma shows that, in the case of Lagrangian torus bundles, Definitions 3.1.3 and Definition 3.1.5 of the parallel transport are equivalent.

Lemma 3.1.7. *Let $F\colon M \to B$ be a Lagrangian torus bundle and $\gamma\colon [0,1] \to B$ be a continuous path. Set*

$$X = \{(x,t) \in M \times [0,1] \colon F(x) = \gamma(t)\}. \tag{3.1}$$

Then $(\alpha_0, -\alpha_1) \in \partial_*(H_2(X, \partial X))$ if and only if the cycle α_1 is a parallel transport of α_0 in the sense of Definition 3.1.3.

Proof. By homotopy invariance, we can assume that γ is smooth and regular. Let $t_0 \le \ldots \le t_n$ be a sufficiently fine partition of $[0,1]$. Then, for each i, we have

$$\gamma([t_i, t_{i+1}]) \subset V_i,$$

where $V_i \subset B$ is a small open neighborhood. By the Arnol'd-Liouville theorem, the two notions of parallel transport along $\gamma|_{[t_i,t_{i+1}]}$ coincide. The result follows. □

Remark 3.1.8. Assume, in addition, that γ is a simple curve. If $\gamma(0) \neq \gamma(1)$, then the manifold X in (3.1) is homeomorphic to $F^{-1}(\gamma)$. If $\gamma(0) = \gamma(1)$, then the manifold X is obtained from $F^{-1}(\gamma)$ by cutting along the fiber $F^{-1}(\xi_0)$.

Following the works [79] and [35], we define fractional monodromy as follows. Consider a (possibly singular) Lagrangian fibration $F\colon M \to B$. In what follows we assume that the map F is proper, so that we have a (singular) torus fibration. Let $\gamma = \gamma(t)$ be a continuous closed curve in $F(M)$ such that the space

$$X = \{(x,t) \in M \times [0,1] \colon F(x) = \gamma(t)\}$$

is connected and such that $\partial X = X_0 \sqcup X_1$ is a disjoint union of two regular tori $X_0 = F^{-1}(\gamma(0))$ and $X_1 = F^{-1}(\gamma(1))$. Set

$$H_1^0 = \{\alpha_0 \in H_1(X_0) \mid \alpha_0 \text{ can be parallel transported along } X\}.$$

Definition 3.1.9. If the parallel transport along X defines an automorphism of the group H_1^0, then this automorphism is called *fractional monodromy along* γ.

Remark 3.1.10. We note that the notion of the parallel transport in the sense of Definition 3.1.3 is not defined when γ crosses critical values of F. Instead, the more general Definition 3.1.5 is used.

Since the work [79], fractional monodromy has been found in various examples of m:$(-n)$ *resonant systems* [35, 78, 86, 91]. It was observed by Bolsinov *et al.* [10] that in such systems the circle action defines a Seifert fibration on a small 3-sphere around the equilibrium point and that the Euler number of this fibration is equal to the number appearing in the corresponding matrix of fractional monodromy. The question that remained unresolved is why this equality holds.

We note that in Chapter 2 we have essentially answered this question for 1:(-1) resonant systems — in such systems we typically have Hamiltonian monodromy. In the present chapter we give a complete answer to the question by proving the following results; see Sections 3.3 and 3.4.

(i) Fractional monodromy can be naturally defined for closed Seifert manifolds (with an orientable base of genus $g > 0$).

(ii) Fractional monodromy is determined by the deck group and the Euler number of the associated Seifert fibration.

(iii) In the case of integrable systems, the Euler number can be computed in terms of the fixed points of the circle action.

The results (ii) and (iii) generalize our results of the previous Chapters 1 and 2 and, in particular, Theorem 2.2.6, thus demonstrating that for both Hamiltonian and fractional monodromy the circle action is more important than the precise form of the integral map F. Together with the result (i), this will allow us to both define and compute fractional monodromy for a much larger class of m:$(-n)$ resonant systems and also for other systems where it has not been observed before; see Section 3.5.

We note that the notion of a deck group was important also for the work [35], where it was defined for a different covering map.

The importance of Seifert fibrations in integrable systems was discovered by Fomenko and Zieschang in the 1980's. In their classification theorem [11,49] Seifert fibrations play a central role: regular isoenergy surfaces of integrable nondegenerate systems with 2 degrees of freedom admit decomposition into families, each of which has a natural structure of a Seifert fibration. In the case of a global circle action there is only one such family, which has a certain label associated to it, the so-called n-mark [10, 11]. We note that the n-mark coincides with the Euler number that appears in Theorem 2.2.6 and is related to this number in the general case; see Remark 3.2.6. Our results in this chapter therefore show how exactly the n-mark determines fractional monodromy.

3.2 1:(−2) resonant system

Here, as a preparation to the more general setting of Sections 3.3 and 3.4, we discuss the famous example of a Hamiltonian system with fractional monodromy due to Nekhoroshev, Sadovskií and Zhilinskií [79].

Consider $\mathbb{R}^4$ with the standard symplectic structure $\omega = dq \wedge dp$. Define the *energy* by

$$H = 2q_1p_1q_2 + (q_1^2 - p_1^2)p_2 + R^2,$$

where $R = \frac{1}{2}(q_1^2 + p_1^2) + (q_2^2 + p_2^2)$, and the *momentum* by

$$J = \frac{1}{2}(q_1^2 + p_1^2) - (q_2^2 + p_2^2).$$

A straightforward computation shows that the functions H and J Poisson commute, so that the map $F = (H, J) \colon \mathbb{R}^4 \to \mathbb{R}^2$ defines an integrable Hamiltonian system. Moreover, the function J defines a Hamiltonian circle action on $\mathbb{R}^4$ which preserves this system.

The bifurcation diagram of the map F is depicted in Figure 3.2. From the structure of the diagram we observe that the Hamiltonian monodromy is trivial. Indeed, the set

$$R = \{\xi \in \operatorname{image}(F) \mid \xi \text{ is a regular value of } F\}$$

is contractible. In particular, every closed path in R can be deformed to a constant path within R. Non-triviality appears if one considers the closed curve γ that is shown in Fig. 3.2. Specifically, there is the following result.

Theorem 3.2.1. ([79]) *Let (a_0, b_0) be an integer basis of $H_1(F^{-1}(\gamma(t_0)))$, where $\gamma(t_0) \in R$ and b_0 is an orbit of the circle action. The fractional monodromy along the curve γ is given by*

$$2a_0 \mapsto 2a_0 + b_0, \;\; b_0 \mapsto b_0.$$

In particular, the parallel transport is unique and H_1^0 is spanned by $2a_0$ and b_0.

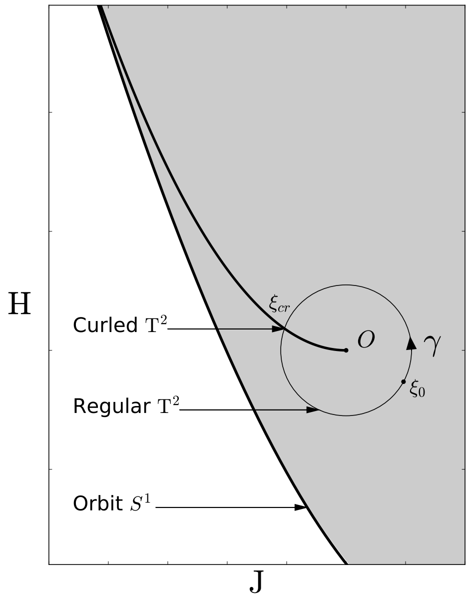

Figure 3.2: The bifurcation diagram of the 1:(−2) resonant system. Critical values are colored black; the set R is shown gray; the closed curve γ around the origin intersects the hyperbolic branch of critical values once and transversely.

Remark 3.2.2. *(Matrix of fractional monodromy)* When written formally in the integer basis (a_0, b_0), the parallel transport has the form of the *rational* matrix

$$\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Q}),$$

called the matrix of *fractional* monodromy.

Since the pioneering work [79], various proofs of Theorem 3.2.1 appeared; see [14, 36, 91, 93] and [35]. Below, as a preparation to Sections 3.3 and 3.4, we give a new proof of this theorem, which is based on the singularities of the circle action. Our proof shows that

- the fixed point $\mathbf{0} \in \mathbb{R}^4$ of the circle action given by J and
- the deck group $\mathbb{Z}_2$ associated to the action

manifest the presence of fractional monodromy in this 1:(−2) resonant system. Later we show that a similar kind of result holds in a general setting of Seifert

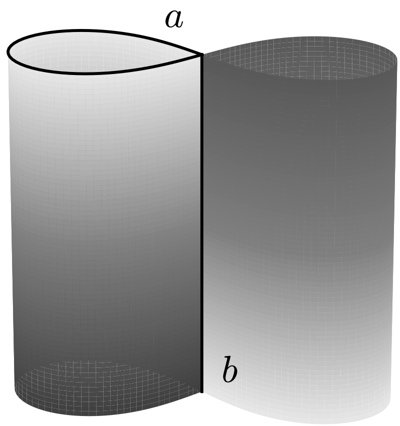

Figure 3.3: Cycles (a, b)

manifolds; see Section 3.3, and, in particular, in the setting of Hamiltonian systems with m:$(-n)$ resonance; see Section 3.5.1.

Proof of Theorem 3.2.1. In complex coordinates $z = p_1 + iq_1$ and $w = p_2 + iq_2$ the circle action given by the momentum J has the form

$$(t, z, w) \mapsto (e^{it}z, e^{-2it}w), \ t \in \mathbb{S}^1. \tag{3.2}$$

We note that the origin is fixed under this action and that the punctured plane

$$P = \{(q, p) \mid q_1 = p_1 = 0 \text{ and } q_2^2 + p_2^2 \neq 0\}$$

consists of points with $\mathbb{Z}_2$ isotropy group. This implies that the Euler number of the Seifert 3-manifold $F^{-1}(\gamma)$, where γ is as in Fig. 3.2, equals $1/2 \neq 0$. Indeed, Stokes' theorem implies that the Euler number of $F^{-1}(\gamma)$ coincides with the Euler number of a small 3-sphere around the origin $z = w = 0$. The latter Euler number equals $1/2$ because of (3.2).

Lemma 3.2.3. *Let $\mathbb{Z}_2 = \{1, -1\}$ denote the order two deck subgroup of the acting circle. The quotient space $F^{-1}(\gamma)/\mathbb{Z}_2$ is the total space of a torus bundle over γ. Its monodromy is given by*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

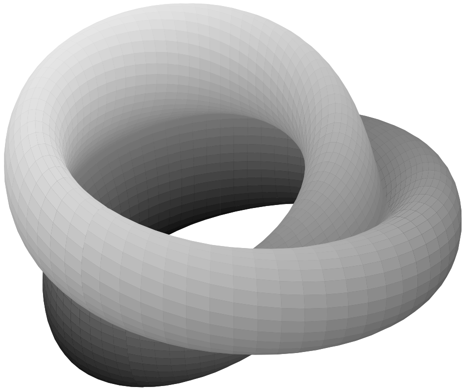

Figure 3.4: Curled torus

Proof. Let $\xi \in \gamma \cap R$. Then the fiber $F^{-1}(\xi)$ is a 2-torus. Since the $\mathbb{S}^1$ action is free on this fiber, the quotient $F^{-1}(\xi)/\mathbb{Z}_2$ is a 2-torus as well.

Consider the critical value $\xi_{cr} \in \gamma$. Its preimage $F^{-1}(\xi_{cr})$ is the so-called *curled torus*; see Fig. 3.4.

Remark 3.2.4. *(Representation of a curled torus)* Take a cylinder over the figure 'eight', as shown in Fig. 3.3. Glue the upper and lower halves of this cylinder after rotating the upper part by the angle π. The resulting singular surface is a curled torus, show in Fig. 3.4.

In this case there is a 'short' orbit b of the $\mathbb{S}^1$ action, formed by the fixed points of the $\mathbb{Z}_2$ action. The 'short' orbit passes through the tip of the cycle a; see Fig. 3.3. Other orbits are 'long', that is, principal. From this description it follows that after taking the $\mathbb{Z}_2$ quotient only half of the cylinder survives and, thus,

$$F^{-1}(\xi_{cr})/\mathbb{Z}_2$$

is topologically a 2-torus. From the structure of the neighborhood of the curled torus in $F^{-1}(\gamma)$, that is, from the description of the A^* atom [49], it follows that

$$F\colon F^{-1}(\xi)/\mathbb{Z}_2 \to \gamma$$

is a torus bundle. In order to complete the proof of Lemma 3.2.3, it is left to apply Theorem 2.2.6. Indeed, since the Euler number of $F^{-1}(\gamma)$ equals 1/2, the Euler number of $F^{-1}(\gamma)/\mathbb{Z}_2$ equals 1. $\square$

Remark 3.2.5. We note that the symplectic form on a neighborhood O of $F^{-1}(\gamma)$ does not descend to the $\mathbb{Z}_2$-quotient space. Therefore, the fibration $F\colon O/\mathbb{Z}_2 \to \mathbb{R}^2$ does not carry a natural Lagrangian structure. In particular, the parallel transport in the sense of Definition 3.1.3 is not defined. Instead, the topological definition of monodromy is used; see Remark 1.1.8 and Definition 3.1.5.

From Lemma 3.2.3 we infer that the parallel transport along the curve γ in the $\mathbb{Z}_2$-quotient space has the form

$$a^r_0 \mapsto a^r_0 + b^r_0,\ b^r_0 \mapsto b^r_0,$$

where the cycles $a^r_0 = a_0/\mathbb{Z}_2$ and $b^r_0 = b_0/\mathbb{Z}_2$ from the induced basis of the group $H_1(F^{-1}(\gamma(t_0))/\mathbb{Z}_2)$. Observe that a_0 is not affected by the quotient map, whereas the orbit b_0 becomes 'shorter': $2b^r_0 \simeq b_0$. It follows that the parallel transport in the original space has the form

$$2a_0 \mapsto 2a_0 + b_0,\ b_0 \mapsto b_0.$$

The results, Theorem 3.2.1, follows. □

Remark 3.2.6. *(Fomenko-Zieschang theory)* Lemma 3.2.3 can be reformulated by saying that the n-mark of the loop molecule associated to γ is equal to 1. The molecule has the form shown in Fig. 3.5. Note that the A^* atom corresponds to the curled torus, Fig. 3.4. A similar statement holds for higher-order resonances. Interestingly, one can change the marks (reglue the fibers) of the loop molecule associated to γ in such a way that fractional monodromy (a) is not defined or (b) is still defined, but the molecule does not admit a global circle action.

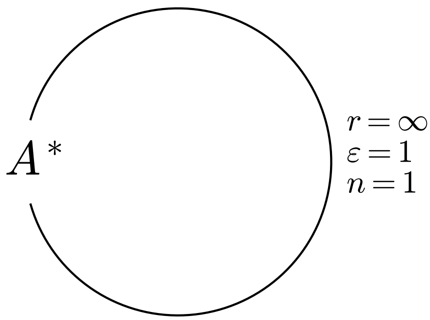

Figure 3.5: The loop molecule associated to γ.

3.3 Parallel transport along Seifert manifolds

In this section we study parallel transport along Seifert manifolds. The obtained results will be applied to fractional monodromy in Section 3.4.

3.3.1 Seifert fibrations

We start with recalling the notions of a Seifert fibration and its Euler number. For a more detailed exposition we refer to [48].

Definition 3.3.1. Let X be a compact orientable 3-manifold (closed or with boundary) which is invariant under an effective fixed point free $\mathbb{S}^1$ action. Assume that the $\mathbb{S}^1$ action is free on the boundary ∂X. Then

$$\rho\colon X \to B = X/\mathbb{S}^1$$

is called a *Seifert fibration*. The manifold X is called a *Seifert manifold*.

Remark 3.3.2. From the slice theorem [4, Theorem I.2.1] (see also [9]) it follows that the quotient $B = X/\mathbb{S}^1$ is an orientable topological 2-manifold. Seifert fibrations are also defined in a more general setting when the base B is non-orientable; see [48, 56]. However, in this case there is no $\mathbb{S}^1$ action and the parallel transport is not unique; see Remark 3.3.7. We will thus consider the orientable case only.

Consider a Seifert fibration

$$\rho\colon X \to B = X/\mathbb{S}^1$$

of a closed Seifert manifold X. Let N be the least common multiple of the orders of the exceptional orbits, that is, the orders of non-trivial isotropy groups. Since X is compact, the number N is well defined. Denote by $\mathbb{Z}_N$ the order N subgroup of the acting circle $\mathbb{S}^1$. The subgroup $\mathbb{Z}_N$ acts on the Seifert manifold X. We have the branched covering map $h\colon X \to X' = X/\mathbb{Z}_N$, the subgroup $\mathbb{Z}_N$ being the deck group of this covering, and the commutative diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\rho"'] & X' \arrow[dl, "\rho'"] \\ B & \end{tikzcd}$$

with ρ' defined via $\rho = \rho' \circ h$. By the construction, $\rho'\colon X' \to B$ is a principal circle bundle over B. We denote its Euler number by $e(X')$.

Definition 3.3.3. The *Euler number* of the Seifert fibration $\rho\colon X \to B = X/\mathbb{S}^1$ is defined by $e(X) = e(X')/N$.

Remark 3.3.4. We note that a closed Seifert manifold X can have non-isomorphic $\mathbb{S}^1$ actions with different Euler numbers. Indeed, let m and n be co-prime integers. Consider the $\mathbb{S}^1$ action

$$(t, z, w) \mapsto (e^{imt}z, e^{-int}w),\ t \in \mathbb{S}^1,$$

on the 3-sphere $S^3 = \{(z, w) \mid |z|^2 + |w|^2 = 1\}$. Then the Euler number of the fibration $\rho\colon S^3 \to S^3/\mathbb{S}^1$ equals $1/mn$. Despite this non-uniqueness, we sometimes refer to $e(X)$ as the Euler number of the Seifert manifold X. This should not be a cause of confusion since it will be always clear from the context what is the underlying $\mathbb{S}^1$ action.

Below we show that the Euler number of a Seifert fibration is an obstruction to the existence of a trivial parallel transport; see Definition 3.1.3.

3.3.2 Parallel transport

Consider a Seifert fibration $\rho\colon X \to B = X/\mathbb{S}^1$ such that the boundary $\partial X = X_0 \sqcup X_1$ consists of two 2-tori X_0 and X_1. Take an orientation and fiber preserving homeomorphism $f\colon X_0 \to X_1$. Any homology basis (a_0, b_0) of $H_1(X_0)$ can be then mapped to the homology basis

$$(a_1 = f_\star(a_0), b_1 = f_\star(b_0))$$

of $H_1(X_1)$. In what follows we assume that b_0 is equal to the homology class of a (any) fiber of the Seifert fibration on X_0. Let

$$X(f) = X/\sim, \quad X_0 \ni x_0 \sim f(x_0) \in X_1,$$

be the closed Seifert manifold that is obtained from X by gluing the boundary components using f.

Finally, let N be the least common multiple of n_j — the orders of the exceptional orbits. With this notation we have the following result.

Theorem 3.3.5. *The parallel transport along X is unique. Only linear combinations of Na_0 and b_0 can be parallel transported along X and under the parallel transport*

$$\begin{aligned} Na_0 &\mapsto Na_1 + kb_1 \\ b_0 &\mapsto b_1 \end{aligned}$$

for some integer $k = k(f)$, which depends only on the isotopy class of f. Moreover, the Euler number of $X(f)$ is given by $e(f) = k(f)/N$.

Proof of Theorem 3.2.1. See Section 3.6. □

Remark 3.3.6. We note that (by the construction) $X(f)/\mathbb{S}^1$ has genus $g > 0$ and hence is not a sphere. It follows that the $\mathbb{S}^1$ action on X and $X(f)$ is unique up to isomorphism; see [52, Theorem 2.3].

Remark 3.3.7. Even if the base B is non-orientable, the group $\partial_*(H_2(X, \partial X))$ is still isomorphic to $\mathbb{Z}^2$. However, in this case, $\partial_*(H_2(X, \partial X))$ is spanned by (b_0, b_1) and $(2b_0, 0)$. It follows that no multiple of a_0 can be parallel transported along X and that the parallel transport is not unique; cf. Remark 3.2.6.

3.3.3 The case of equivariant filling

Theorem 3.3.5 shows that the Euler number of a Seifert manifold can be computed in terms of the parallel transport along this manifold. But conversely, if we know the Euler number and the orders of exceptional orbits of a Seifert manifold, we also know how the parallel transport acts on homology cycles. In applications the orders of exceptional orbits are often known. In order to compute the Euler number one may then use the following result.

Theorem 3.3.8. *Let M be a compact oriented 4-manifold that admits an effective circle action. Assume that the action is fixed-point free on the boundary ∂M and has only finitely many fixed points $p_1, \ldots, p_\ell$ in the interior. Then*

$$e(\partial M) = \sum_{k=1}^{\ell} \frac{1}{m_k n_k},$$

where (m_k, n_k) are isotropy weights of the fixed points p_k.

Remark 3.3.9. Recall that near each fixed point p_k the $\mathbb{S}^1$ action can be linearized as

$$(t, z, w) \mapsto (e^{im_k t} z, e^{-in_k t} w), \ t \in \mathbb{S}^1, \tag{3.3}$$

in appropriate coordinates (z, w) that are positive with respect to the orientation of M. The isotropy weights m_k and n_k are co-prime integers. In particular, none of them is equal to zero.

Remark 3.3.10. In the above theorem neither M nor ∂M are assumed to be connected. The orientation on ∂M is induced by M.

Proof of Theorem 3.3.8. Eq. (3.3) implies that for each fixed point p_k there exists a small closed 4-ball $B_k \ni p_k$ invariant under the action. Denote by Z the manifold $Z = M \setminus \bigcup_{k=1}^{\ell} B_k$. Let N be a common multiple of the orders of all exceptional orbits in M and $\mathbb{Z}_N$ be the order N subgroup of the acting circle $\mathbb{S}^1$. Set

$$X = Z/\mathbb{Z}_N \quad \text{and} \quad Y = Z/\mathbb{S}^1.$$

Denote by $\mathrm{Pr}\colon X \to Y$ the natural projection that identifies the orbits of the $\mathbb{S}^1/\mathbb{Z}_N$ action. By the construction the triple (X, Y, Pr) is a principal circle bundle.

Because of the slice theorem [4] the spaces X and Y are topological manifolds (with boundaries). The boundary ∂Y is a disjoint union of the closed 2-manifold $B = \partial M/\mathbb{S}^1$ and the 2-spheres $S_k^2 = \partial B_k/\mathbb{S}^1$. Let $i_B\colon B \to Y$ and $i_k\colon S_k^2 \to Y$ be the corresponding inclusions.

Denote by $\mathbf{e}_Y \in H^2(Y)$ the Euler class of the circle bundle (X, Y, Pr). By the functoriality $i_B^*(\mathbf{e}_Y)$ and $i_k^*(\mathbf{e}_Y)$ are the Euler classes of the circle bundles $(\mathrm{Pr}^{-1}(B), B, \mathrm{Pr})$ and $(\mathrm{Pr}^{-1}(S_k^2), S_k^2, \mathrm{Pr})$, respectively. Hence

$$\langle \mathbf{e}_Y, i_B(B)\rangle = \langle i_B^*(\mathbf{e}_Y), B\rangle = Ne(\partial M)$$

and analogously

$$\langle \mathbf{e}_Y, i_k(S_k^2)\rangle = \langle i_k^*\mathbf{e}_Y, S_k^2\rangle = \frac{N}{m_k n_k}.$$

The equality

$$\langle \mathbf{e}_Y, i_B(B) - \sum_{k=1}^{\ell} i_k(S_k^2)\rangle = \langle \mathbf{e}_Y, \partial Y\rangle = 0$$

completes the proof. □

3.4 Applications to integrable systems

Consider a singular Lagrangian fibration $F\colon M \to R$ over a 2-dimensional manifold R. Assume that the map F is proper and invariant under an effective $\mathbb{S}^1$ action. Take a simple closed curve $\gamma = \gamma(t)$ in $F(M)$ that satisfies the following regularity conditions:

(i) the fiber $F^{-1}(\gamma(0))$ is regular and connected;

(ii) the $\mathbb{S}^1$ action is fixed-point free on the preimage $E = F^{-1}(\gamma)$;

(iii) the preimage E is a closed oriented connected submanifold of M.

Remark 3.4.1. Note that, generally speaking, $F^{-1}(\gamma(t))$, $t \in [0, 1]$, is neither smooth nor connected.

From the regularity conditions it follows that

$$X = \{(x, t) \in M \times [0, 1]\colon F(x) = \gamma(t)\}$$

is a Seifert manifold with an orientable base. This manifold can be obtained from the Seifert manifold $E = F^{-1}(\gamma)$ by cutting along the fiber $F^{-1}(\gamma(0))$. We note that the boundary $\partial X = X_0 \sqcup X_1$ is a disjoint union of two tori.

Let $e(E)$ be the Euler number of E and N denote the least common multiple of n_j — the orders of the exceptional orbits. Take a basis (a, b) of the homology group $H_1(X_0) \simeq \mathbb{Z}^2$, where b is given by any orbit of the $\mathbb{S}^1$ action. Then the following theorem holds.

Theorem 3.4.2. *Fractional monodromy along γ is defined. Moreover, (Na, b) form a basis of the parallel transport group H_1^0 and the corresponding isomorphism has the form $b \mapsto b$ and $Na \mapsto Na + kb$, where $k \in \mathbb{Z}$ is given by $k = Ne(E)$.*

Proof. Follows directly from Theorem 3.3.5. □

Remark 3.4.3. Theorem 3.4.2 tells us that the Euler number $e(E)$ and the least common multiple N completely determine fractional monodromy along γ.

Remark 3.4.4. Let $i_0 \colon X_0 \to X$ and $i_1 \colon X_1 \to X$ denote the corresponding inclusions. Observe that, in our case, the composition

$$i_1^{-1} \circ i_0 \colon H_1(X_0, \mathbb{Q}) \to H_1(X_0, \mathbb{Q})$$

gives an automorphism of the first homology group $H_1(X_0, \mathbb{Q})$. In a basis of $H_1(X_0, \mathbb{Z})$ the isomorphism $i_1^{-1} \circ i_0$ is written as 2×2 matrix with rational coefficients, called the *matrix of fractional monodromy* [93]. We have thus proved that in a basis (a, b) of $H_1(M_0)$, where b corresponds to the $\mathbb{S}^1$ action, the fractional monodromy matrix has the form

$$\begin{pmatrix} 1 & e(E) = k/N \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Q}).$$

In certain cases we can easily compute the parameter $e(E) = k/N$, as is explained in the following theorem.

Theorem 3.4.5. *Assume that γ bounds a compact 2-manifold $U \subset R$ such that $F^{-1}(U)$ has only finitely many fixed points $p_1, \ldots, p_l$ of the $\mathbb{S}^1$ action. Then*

$$e(E) = \sum_{k=1}^{l} \frac{1}{m_k n_k},$$

where (m_k, n_k) are the isotropy weights of the fixed points p_k.

Proof. Follows directly from Theorem 3.3.8. □

Remark 3.4.6. For the case of standard monodromy, Theorem 3.4.5 agrees with Theorem 2.2.14, which considers only the case $m_k = 1$ and $n_k = \pm 1$ and which states that the monodromy parameter is given by the sum of positive singular points $(n_k = 1)$ of the Hamiltonian $\mathbb{S}^1$ action minus the number of negative singular points $(n_k = -1)$.

Remark 3.4.7. Theorem 3.3.8, when applied to the context of singular Lagrangian fibrations, tells us more than Theorem 3.4.5. Indeed, consider smooth curves γ_1 and γ_2 that are cobordant in R. Theorem 3.3.8 allows to compute

$$e(F^{-1}(\gamma_1)) - e(F^{-1}(\gamma_2)),$$

which is the difference between the Euler numbers of $F^{-1}(\gamma_1)$ and $F^{-1}(\gamma_2)$. This difference shows how far is fractional monodromy along γ_1 from fractional monodromy along γ_2. Theorem 3.4.5 is recovered when γ_1 is cobordant to zero.

Combining Theorems 3.4.2 and 3.4.5 together one can compute fractional monodromy in various integrable Hamiltonian systems. We illustrate this in the following Section 3.5.

3.5 Examples

3.5.1 Resonant systems

In this section we consider $m{:}(-n)$ *resonant systems* [35,78,86,91], which are local models for integrable 2 degrees of freedom systems with an effective Hamiltonian $\mathbb{S}^1$ action. Our approach to these systems is very general. Moreover, it clarifies a question posed in [10, Problem 61], cf. Remark 3.2.6.

Definition 3.5.1. Consider $\mathbb{R}^4$ with the canonical symplectic structure $dq \wedge dp$. An integrable Hamiltonian system

$$(\mathbb{R}^4, dq \wedge dp, F = (J, H))$$

is called a $m{:}(-n)$ *resonant system* if the function J is the $m{:}(-n)$ *oscillator*

$$J = \frac{m}{2}(q_1^2 + p_1^2) - \frac{n}{2}(q_2^2 + p_2^2).$$

Here m and n be relatively prime integers with $m > 0$.

We note that for every $m{:}(-n)$ resonant system there exists an associated effective $\mathbb{S}^1$ action that preserves the integral map $F = (J, H)$. Indeed, the induced Hamiltonian flow of J is periodic. In coordinates $z = p_1 + iq_1$ and $w = p_2 + iq_2$ the action has the form

$$(t, z, w) \mapsto (e^{imt}z, e^{-int}w), \ t \in \mathbb{S}^1. \tag{3.4}$$

Assume that the integral map $F = (J, H)$ is proper. Let $\gamma = (J(t), H(t))$ be a simple closed curve satisfying the assumptions (i)-(iii) from Section 3.4.

Remark 3.5.2. We note that, in this case, the assumptions (i)-(iii) can be reduced to the following more easily verifiable conditions

(i') the fiber $F^{-1}(\gamma(0))$ is regular and connected;

(ii') the preimage $E = F^{-1}(\gamma)$ is connected;

(iii') for all t the following holds: $H'(t)dJ - J'(t)dH \neq 0$.

Proof. Under (i')-(iii'), the space $E = F^{-1}(\gamma)$ is the boundary of the compact oriented manifold $F^{-1}(U)$, where U is the 2-disk bounded by the curve γ. Hence, E is itself compact and oriented. It is left to note that the $\mathbb{S}^1$ action is fixed-point free on E. $\square$

Let (a, b) be a basis of the integer homology group $H_1(F^{-1}(\gamma(0))$ such that b is given by any orbit of the $\mathbb{S}^1$ action. There is the following result (cf. [35]).

Theorem 3.5.3. *Let U be the 2-disk in the (J, H)-plane that is given by $\partial U = \gamma$. If $(0, 0) \in U$, then the parallel transport group H_1^0 is spanned by mna and b. The matrix of fractional monodromy has the form*

$$\begin{pmatrix} 1 & 1/mn \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Q}).$$

If $(0, 0) \notin U$, then the parallel transport group H_1^0 is spanned by Na and b, where $N \in \{1, m, n, mn\}$. The matrix of fractional monodromy is trivial.

Proof. In view of Theorems 3.4.2 and 3.4.5, we only need to determine the least common multiple N.

Let $(0, 0) \in U$. In this case the fixed point $q = p = 0$ of the $\mathbb{S}^1$ action belongs to $F^{-1}(U) \subset \mathbb{R}^4$. Orbits with $\mathbb{Z}_m$ and $\mathbb{Z}_n$ isotropy group emanate from this fixed point and necessarily 'hit' the boundary $F^{-1}(\gamma)$. It follows that the least common multiple is $N = mn$.

Let $(0, 0) \notin U$. In this case the fixed point $q = p = 0$ of the $\mathbb{S}^1$ action does not belong to $F^{-1}(U) \subset \mathbb{R}^4$. However, γ might intersect critical values of F that give rise to exceptional orbits in $E = F^{-1}(\gamma)$ with $\mathbb{Z}_m$ or $\mathbb{Z}_n$ isotropy group. It follows that the least common multiple is $N = 1$, m, n or mn. $\square$

Remark 3.5.4. If $mn < 0$, then the fixed point $z = w = 0$ of the $\mathbb{S}^1$ action is necessarily at the boundary of the corresponding bifurcation diagram. Hence non-trivial monodromy (standard or fractional) can only be found when $mn > 0$. Because of Theorem 3.5.3, non-trivial standard monodromy can manifest itself only when $m = n = 1$.

Example 3.5.5. An example of such a 1:-1 resonant system can be obtained by considering the Hamiltonian

$$H = p_1 q_2 + p_2 q_1 + \varepsilon(q_1^2 + p_1^2)(q_2^2 + p_2^2).$$

The bifurcation diagram of the integral map $F = (J, H)$ has the form shown in Fig. 3.6. From Theorem 3.5.3 we infer that the monodromy matrix along γ has the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

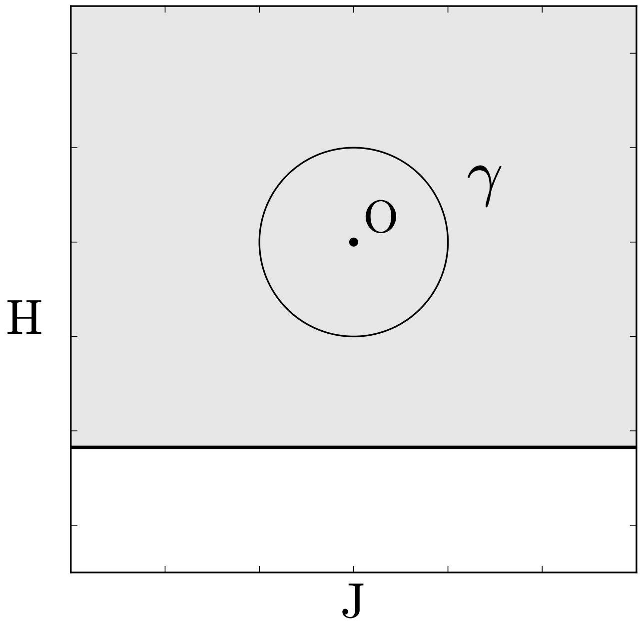

Figure 3.6: Bifurcation diagram of a 1:(−1) system. The set of regular values is shown gray; the critical values are colored black; the isolated critical point $O = (0, 0)$ lifts to the singly pinched torus $F^{-1}(O)$.

Example 3.5.6. An example of a m:($-n$) resonant system with non-trivial fractional monodromy is the specific 1:(−2) resonant system, which has been introduced in [79]. The system is obtained by considering the Hamiltonian

$$H = 2q_1p_1q_2 + (q_1^2 - p_1^2)p_2 + \varepsilon R(q, p)^2,$$

where $\varepsilon > 0$ and $R = R(q, p)$ is the 1:(2) oscillator. The bifurcation diagram of the integral map $F = (J, H)$ has the form shown in Fig. 3.2. In this case the set of regular values is simply connected and, thus, standard monodromy is trivial. Let the curve γ be as in Fig. 3.2. From Theorem 3.5.3 we infer that the parallel transport group H_1^0 is spanned by $2a$ and b, and that the fractional monodromy matrix has the form

$$\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Q}).$$

This system is discussed in greater detail in Section 3.2.

3.5.2 A system on $S^2 \times S^2$

Let (x_1, x_2, x_3) and (y_1, y_2, y_3) be coordinates in $\mathbb{R}^3$. The relations

$$\{x_i, x_j\} = \epsilon_{ijk} x_k, \ \{y_i, y_j\} = \epsilon_{ijk} y_k \text{ and } \{x_i, y_j\} = 0$$

define a Poisson structure on $\mathbb{R}^3 \times \mathbb{R}^3$. The restriction of this Poisson structure to $S^2 \times S^2 = \{(x, y) : |x| = |y| = 1\}$ gives the canonical symplectic structure ω.

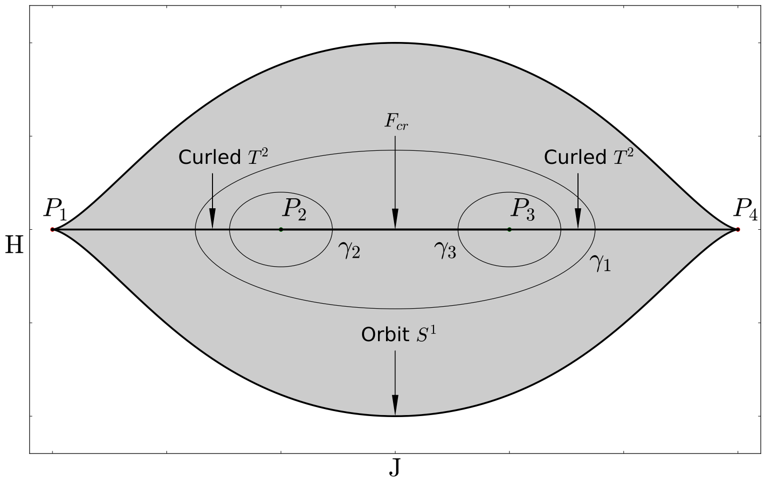

Figure 3.7: Bifurcation diagram of the integral map F. The set of regular values shown gray; the critical values are colored black. All regular fibers are 2-tori. Curled T^2 contains one exceptional ('short') orbit of the $\mathbb{S}^1$ action. Critical fibers F_{cr} contain two such orbits. They can be obtained by gluing two curled tori along a regular orbit of the $\mathbb{S}^1$ action.

We consider an integrable Hamiltonian system on $(S^2 \times S^2, \omega)$ defined by the integral map $F = (J, H) : S^2 \times S^2 \to \mathbb{R}^2$, where

$$J = x_1 + 2y_1 \quad \text{and} \quad H = \mathrm{Re}\{(x_2 + ix_3)^2 (y_2 - iy_3)\}.$$

It is easily checked that the functions J and H commute, so F is indeed an integral map. The bifurcation diagram is shown in Fig. 3.7.

Even without knowing the precise structure of critical fibers of F, we can compute fractional monodromy along curves γ_1, γ_2 and γ_3, shown in Fig. 3.7. Specifically, assume that $\gamma_i(0) = \gamma_i(1)$ lifts to a regular torus.

Theorem 3.5.7. *For each γ_i, the parallel transport group is spanned by $2a_i$ and b_i, where (a_i, b_i) forms a basis of $H_1(F^{-1}(\gamma_i(0))$ and b_i is given by any orbit of* *the $\mathbb{S}^1$ action. The fractional monodromy matrices have the form*

$$\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \text{ for } i = 2, 3 \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ for } i = 1.$$

Proof. Consider the case $i = 2$. The other cases can be treated analogously. The curve γ_2 intersects the critical line $H = 0$ at two points ξ_1 and ξ_2. Let $\xi_1 < P_2 < \xi_2$ on $H = 0$. The critical fiber $F^{-1}(\xi_1)$, which is a curled torus, contains one exceptional orbit of the $\mathbb{S}^1$ action with $\mathbb{Z}_2$ isotropy. The critical fiber $F^{-1}(\xi_1)$ contains two such orbits. Finally, observe that the point

$$(1, 0, 0) \times (-1, 0, 0) \in S^2 \times S^2,$$

which projects to P_2 under the map F, is fixed under the $\mathbb{S}^1$ action and has isotropy weights $m = 1, n = 2$. Since $F^{-1}(\gamma_2)$ is connected, it is left to apply Theorems 3.4.2 and 3.4.5. □

3.5.3 Revisiting the quadratic spherical pendulum

The example of the system on $S^2 \times S^2$ discussed above shows that the fractional monodromy matrix along a given curve γ_1 could be an integer matrix even if standard monodromy along γ_1 is not defined. Here we show that the same phenomenon can appear when the isotropy groups are either trivial or $\mathbb{S}^1$, that is, when the $\mathbb{S}^1$ action is free outside fixed points. We demonstrate this on quadratic spherical pendulum, which was discussed in Section 2.3.2.

Let S^2 be the unit sphere in $\mathbb{R}^3$ with coordinates (x, y, z). We recall that the Hamiltonian system on T^*S^2 defined by the Hamiltonian function

$$H = \frac{1}{2}\langle p, p\rangle + V(z),$$

where $V(z) = bz^2 + cz$, is called the quadratic spherical pendulum [34]. This system is completely integrable since the z component J of the angular momentum is conserved. Moreover, J generates a global Hamiltonian $\mathbb{S}^1$ action on T^*S^2. For a certain range of b and c, the bifurcation diagram of the integral map $F = (J, H)$ has the form shown in Fig. 3.8.

Let γ_1 and γ_2 be as in Fig. 3.8. Assume that the starting point $\gamma_i(0) = \gamma_i(1)$ lifts to a regular torus.

Theorem 3.5.8. *For each γ_i, the parallel transport group coincides with the whole homology group $H_1(F^{-1}(\gamma_i(0)))$. The fractional monodromy matrices have the form*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ for } i = 1 \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } i = 2.$$

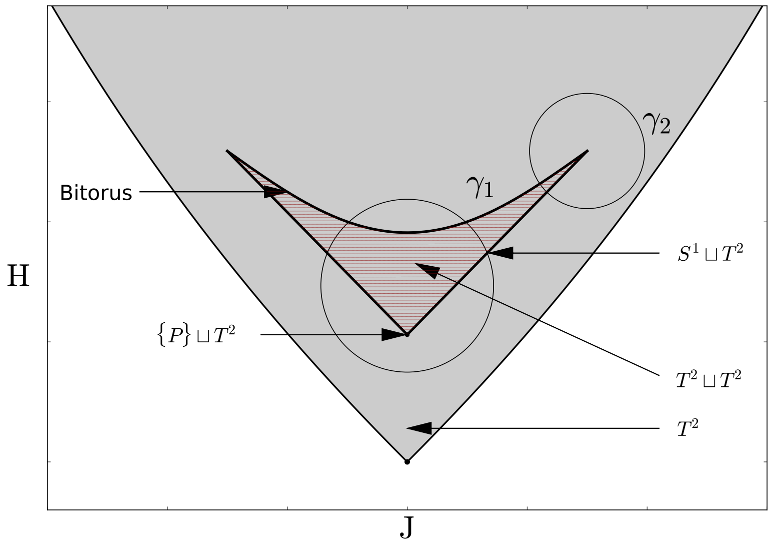

Figure 3.8: Bifurcation diagram of the integral map F. The set of regular values shown gray. The critical values are colored black. The points in the interior of the 'island' are regular and lift to the disjoint union of 2 tori.

Proof. Consider the case $i = 1$. The other case can be treated similarly. The $\mathbb{S}^1$ action is free on the connected manifold $F^{-1}(\gamma_1)$. The Euler number of this manifold equals 1. Indeed, the elliptic-elliptic point

$$P = (0,0,1) \times (0,0,0) \in T^*S^2 \subset T^*\mathbb{R}^3,$$

which projects to the point $F(P) \in \operatorname{int}(\gamma_1)$, is fixed under the $\mathbb{S}^1$ action and has isotropy weights $m = 1, n = 1$. It is left to apply Theorems 3.4.2 and 3.4.5. $\square$

Remark 3.5.9. From Theorem 3.5.8 it follows that all homology cycles can be parallel transported along γ_i, $i = 1, 2$. Even though this situation is very similar to the case of standard monodromy, the monodromy along γ_i is fractional. We note that such examples have not been considered until now.

3.6 Proof of the main theorem

In the present section we use the notation introduced in Section 3.3. The result, that is, Theorem 3.3.5, will follow from Lemmas 3.6.1, 3.6.3, 3.6.4, and 3.6.5 which are given below.

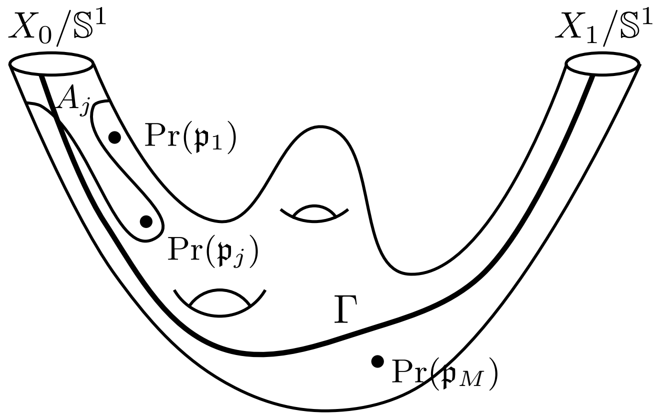

Figure 3.9: The base manifold $X/\mathbb{S}^1$.

Lemma 3.6.1. *There exists $k \in \mathbb{Z}$ such that $(Na_0, Na_1 + kb_1)$ and (b_0, b_1) belong to $\partial_*(H_2(X, \partial X))$.*

Proof. Let $\mathbb{Z}_N$ be the order N subgroup of $\mathbb{S}^1$. The quotient $X' = X/\mathbb{Z}_N$, which is given by the induced action of the subgroup $\mathbb{Z}_N$, is the total space of the principal circle bundle

$$\mathrm{Pr}' \colon X' \to X/\mathbb{S}^1.$$

We note that this bundle is, moreover, trivial. Indeed, the base $X/\mathbb{S}^1$ has a boundary and is, thus, homotopy equivalent to a graph.

Let $b_i^r = b_i/\mathbb{Z}_N,\ i = 0, 1$. Then (a_i, b_i^r) forms a basis of $H_1(X_i/\mathbb{Z}_N)$. There is a unique parallel transport of the cycles a_0 and b_0^r along X'. Indeed, take a global section

$$s \colon X'/\mathbb{S}^1 \to X' \text{ with } s(X_0/\mathbb{S}^1) = a_0.$$

Then $S = s(X'/\mathbb{S}^1)$ is a relative 2-cycle that gives the parallel transport of a_0. In order to transport the cycle b_0^r take a smooth curve $\Gamma \subset X/\mathbb{S}^1$ connecting $X_0/\mathbb{S}^1$ with $X_1/\mathbb{S}^1$ and define the relative 2-cycle by $(\mathrm{Pr}')^{-1}(\Gamma)$.

Remark 3.6.2. In what follows we assume that Γ is a simple curve that does not contain the singular points $\mathrm{Pr}(\mathfrak{p}_1), \ldots, \mathrm{Pr}(\mathfrak{p}_M)$, where $\mathrm{Pr}\colon X \to X/\mathbb{S}^1$ is the canonical projection; see Fig. 3.9.

From above it follows that the parallel transport in the reduced space has the form $a_0 \mapsto a_1 + kb_1^r$ and $b_0^r \mapsto b_1^r$ for some $k \in \mathbb{Z}$. The parallel transport of the cycles (a_0, b_0^r) in the reduced space lifts to the parallel transport of the cycles

(Na_0, b_0) along X in the original space. Indeed, let $\pi\colon X \to X'$ be the quotient map, given by the action of $\mathbb{Z}_N$. The preimage

$$\pi^{-1}((\mathrm{Pr}')^{-1}(\Gamma)) = \mathrm{Pr}^{-1}(\Gamma)$$

transports b since Γ does not contain the singular points $\mathrm{Pr}(\mathfrak{p}_j)$. In order to transport Na take $\pi^{-1}(S)$. Since $\pi\colon \pi^{-1}(S) \to S$ is a branched N-covering, see Fig. 3.10, the preimage $\pi^{-1}(S)$ is a relative 2-cycle that transports Na. The result follows. □

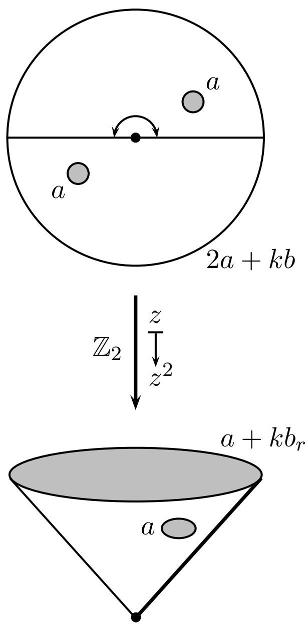

Figure 3.10: An example of the covering map $\pi\colon \pi^{-1}(S) \to S$. Here the Seifert manifold X contains only one exceptional orbit with $\mathbb{Z}_2$ isotropy ($N = 2$); the base $X/\mathbb{S}^1 \cong S$ is a 'cone with a hole'.

This following lemma shows that the parallel transport along X is unique.

Lemma 3.6.3. *Suppose that $(0, c) \in \partial_*(H_2(X, \partial X))$ for some $c \in H_1(X_1)$. Then we have $c = 0$.*

Proof. This statement was essentially proved in [35] (see §7.1 therein). For the sake of completeness we provide a proof below.

Since X is an orientable 3-manifold, the rank of the image $\partial_*(H_2(X, \partial X))$ is half of the rank of $H_1(\partial X) \simeq \mathbb{Z}^2 \oplus \mathbb{Z}^2$. Hence

$$\mathrm{rk}\ \partial_*(H_2(X, \partial X)) = 2.$$

As a subgroup of a free abelian group $H_1(\partial X)$, the image $\partial_*(H_2(X, \partial X))$ is a free abelian group and thus is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

From Lemma 3.6.1 we get that $\alpha = (Na_0, Na_1 + kb_1)$ and $\beta = (b_0, b_1)$ belong to $\partial_*(H_2(X, \partial X))$. Suppose that parallel transport along X is not unique. Then there exists an element

$$\eta = (0, c) \in \partial_*(H_2(X, \partial X))$$

with $c \neq 0$. Since α and β are linearly independent over $\mathbb{Z}$, we get $l_1\alpha + l_2\beta = l_3\eta$, where l_j are integers and $l_3 \neq 0$. But $l_1 Na_0 + l_2 b_0 = 0$, so $l_1 = l_2 = 0$ and we get a contradiction. $\square$

The set H_1^0 of cycles $\alpha \in H_1(X_0)$ that can be parallel transported along X forms a subgroup of $H_1(X_0)$. Since Na_0 and b_0 can be parallel transported along X, the group H_1^0 is spanned by La_0 and b_0 for some $L \in \mathbb{N}$, which divides N. Our goal is to prove that $L = N$. The proof of this equality is based on the important Lemma 3.6.4 below.

Let E be a closed Seifert manifold which is obtained from X by identifying the boundary tori X_i via an orbit preserving diffeomorphism that sends a_0 to a_1 and b_0 to b_1.

Lemma 3.6.4. *The Euler number $e(E)$ of the Seifert manifold E satisfies*

$$e(E) = \frac{k}{N}.$$

Proof. Consider the action of the quotient circle $\mathbb{S}^1/\mathbb{Z}^N$ on the quotient space $E' = E/\mathbb{Z}^N$. Since E' is a manifold and the action is free, we have a principal bundle $(E', B = E/\mathbb{S}^1, \mathrm{Pr}\,')$. Let

$$U_1 \cong [0, \varepsilon] \times S^1$$

be a cylindrical neighborhood of $X_0/\mathbb{S}^1$ in $X/\mathbb{S}^1$ with $\{0\} \times S^1 \cong X_0/\mathbb{S}^1$. Define

$$U_2 = \overline{B \setminus U_1}.$$

We already know that if $X' = X/\mathbb{Z}_N$ then $(X', X/\mathbb{S}^1, \mathrm{Pr}\,')$ is a trivial circle bundle. Observe that E' is obtained from X' by identifying the boundary tori X'_0 and X'_1 via a diffeomorphism induced by the 'monodromy' matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. Hence there exist cross sections $s_1 \colon U_1 \to E'$ and $s_2 \colon U_2 \to E'$ such that $s_2 = s_1$ on the boundary circle $\{\varepsilon\} \times S^1$ and $s_1 = e^{ik\varphi} s_2$ on $\{0\} \times S^1$, where the circle is parametrized by an angle φ.

Let $f \colon [0, 2\pi] \to [0, 1]$ be a smooth function such that $f|_{[0,\delta]} = 1$ and $f|_{[2\delta, 2\pi]} = 0$. Define a continuous function $h \colon [0, \varepsilon] \times S^1 \to [0, 2\delta]$ by the following formula

$$h(\phi, \varphi) = \frac{\varepsilon - \phi}{\varepsilon} \varphi f(\varphi).$$

Let $D^2 = (0, \varepsilon) \times (\delta, 2\pi)$. Define new cross sections $s_1'\colon U_1 \to E'$ and $s_2'\colon B \setminus D^2 \to E'$ as follows

$$s_1' = s_1 \cdot e^{ikh} \quad \text{and} \quad s_2' = \begin{cases} s_2 & \text{on } U_2, \\ s_1' & \text{otherwise.} \end{cases}$$

Observe that $s_1(0 \times S^1) = s_2(0 \times S^1) + kb$, where b corresponds to the $\mathbb{S}^1$ action. If $\delta > 0$ is small enough, then $s_1(0 \times S^1)$ is homological to $s_1'(0 \times S^1)$. Hence

$$s_1'(0 \times S^1) = s_2'(0 \times S^1) + kb.$$

But $s_1'(\partial D^2 + 0 \times S^1) = s_2'(\partial D^2 + 0 \times S^1)$. Therefore

$$s_2'(\partial D^2) = s_1'(\partial D^2) + kb.$$

Thus, $e(E') = k$ and

$$e(E) = \frac{1}{N} e(E') = \frac{k}{N}.$$

□

Lemma 3.6.5. *The parallel transport group H_1^0 is spanned by the cycles Na_0 and b_0.*

Proof. We have already noted that H_1^0 is spanned by La_0 and b_0 for some $L \in \mathbb{N}$, which divides N. In order to prove the equality $L = N$ it is sufficient to prove that for every j the number L is a multiple of n_j (the order of the exceptional orbit $\mathfrak{p}_j$).

The image of the exceptional fiber $\mathfrak{p}_j$ under the projection $\mathrm{Pr}\colon E \to B = E/\mathbb{S}^1$ is a single point $\mathrm{Pr}(\mathfrak{p}_i)$ on the base manifold B. Cutting E along the torus $X_0 \cong X_1$ results in the manifold X. The quotient $X/\mathbb{S}^1$ is obtained from B by cutting along an embedded circle. Consider an annulus $A_j \subset X/\mathbb{S}^1$ that contains $X_0/\mathbb{S}^1$ and exactly one singular point $\mathrm{Pr}(\mathfrak{p}_j)$; see Fig 3.9.

Clearly, the preimage $E_j = \mathrm{Pr}^{-1}(A_j)$ is a Seifert manifold with only one exceptional fiber. From the definition of the parallel transport it follows that there exists a relative cycle $S \subset E_j$ such that one of the connected components of S is La_0. In other words, La_0 can be parallel transported along E_j.

Let us identify the boundary tori of E_j via an orbit preserving diffeomorphism. Then the result of the parallel transport of La_0 along E_j is $l_1 a_0 + l_2 b_0$. Since the parallel transport is unique, see Lemma 3.6.3, we have

$$Nl_1 a_0 + Nl_2 b_0 = NLa_0 = LNa_0 = LNa_0 + Lm_j b_0, \tag{3.5}$$

where $m_j \in \mathbb{Z}$. Let e_j denote the Euler number of the Seifert manifold E_j. From Lemma 3.6.4 it follows that

$$m_j/n_j = e_j \pmod 1.$$

In particular, m_j and n_j are relatively prime. Eq. (3.5) implies $Nl_2 = Lm_j$. Since n_j divides N, it also divides L. □